

# ON THE GENERALISATION OF THE HAHN-JORDAN DECOMPOSITION FOR REAL CÀDLÀG FUNCTIONS

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**ABSTRACT.** For a real càdlàg function  $f$  and a positive constant  $c$  we find another càdlàg function, which has the smallest total variation possible among all functions uniformly approximating  $f$  with accuracy  $c/2$ . The solution is expressed with the truncated variation, upward truncated variation and downward truncated variation introduced in [L1] and [L2]. They are always finite even if the total variation of  $f$  is infinite, and they may be viewed as the generalisation of the Hahn-Jordan decomposition for real càdlàg functions. We also present partial results for more general functions.

## 1. INTRODUCTION

Let  $E$  be a metric space with the metric  $\rho$ ,  $a < b$  be two reals and  $f : [a; b] \rightarrow E$ . We define the total variation of the function  $f$  as

$$TV(f, [a; b]) = \sup_n \sup_{\pi_n} \sum_{i=1}^n \rho(f(t_i), f(t_{i-1})),$$

where the second supremum is over all partitions  $\pi_n = \{a \leq t_0 < t_1 < \dots < t_n \leq b\}$ .

In general, the total variation of  $f$  may be (and in many important cases is) infinite. However, after imposing some regularity conditions on  $f$  we will easily find functions approximating  $f$  with arbitrary accuracy and having finite total variation, even if the total variation of  $f$  is infinite. Obviously, the better approximation is, the greater is the total variation of the approximating function. Let us fix  $c > 0$ . The natural question arises, what is the greatest lower bound for total variation of function  $g : [a; b] \rightarrow E$ , uniformly approximating  $f$  with accuracy  $c/2 > 0$ , i.e. what is

$$\inf_{g \in B(f, c/2)} TV(g, [a; b]),$$

where  $B(f, d)$  denotes the ball

$$B(f, d) := \left\{ g : [a; b] \rightarrow E : \sup_{t \in [a; b]} \rho(f(t), g(t)) \leq d \right\}.$$

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The immediate bound from below for  $\inf_{g \in B(f, c/2)} TV(g, [a; b])$  reads as

$$(1.1) \quad \inf_{g \in B(f, c/2)} TV(g, [a; b]) \geq \sup_n \sup_{\pi_n} \sum_{i=1}^n \max \{ \rho(f(t_i), f(t_{i-1})) - c, 0 \}$$

and follows directly from the triangle inequality

$$\begin{aligned} \rho(g(t_i), g(t_{i-1})) &\geq \rho(f(t_i), f(t_{i-1})) - \rho(f(t_i), g(t_i)) - \rho(f(t_{i-1}), g(t_{i-1})) \\ &\geq \rho(f(t_i), f(t_{i-1})) - c. \end{aligned}$$

We will call the quantity on the right hand side of (1.1), i.e.

$$\sup_n \sup_{\pi_n} \sum_{i=1}^n \max \{ \rho(f(t_i), f(t_{i-1})) - c, 0 \}$$

*truncated variation of the function  $f$  at the level  $c$*  and denote it by  $TV^c(f, [a; b])$ ; it was first introduced in [L1].

The just obtained lower bound for  $\inf_{g \in B(f, c/2)} TV(g, [a; b])$  may also be infinite but it appears to be finite one e.g. for functions which are uniform limits of sequences of step functions. Two important families of such functions are regulated functions (see [D, Sect. VII.6]) and càdlàg functions, i.e. right-continuous functions with left limits.

The family of real càdlàg functions will be of our special interest. It is due to easily proven fact (see Fact 2.1) that any càdlàg function is an uniform limit of a sequence of càdlàg functions with finite total variations, which correspond naturally to finite signed measures on the interval  $(a; b]$ . Moreover, in this paper we will show that for càdlàg  $f$ ,  $E = \mathbb{R}$  with the standard Euclidean metric  $\rho(x, y) = |x - y|$  and  $c > 0$  we have in fact equality, i.e.

$$(1.2) \quad \inf_{\|g-f\|_\infty \leq c/2} TV(g, [a; b]) = TV^c(f, [a; b]),$$

where  $g : [a; b] \rightarrow \mathbb{R}$ ,  $\|g - f\|_\infty = \sup_{t \in [a; b]} |g(t) - f(t)|$ . Moreover, there exist such a càdlàg function  $f^c : [a; b] \rightarrow \mathbb{R}$  that

$$\|f^c - f\|_\infty \leq c/2 \text{ and } TV(f^c, [a; b]) = TV^c(f, [a; b]).$$

**Remark 1.1.** In general, the function  $f^c$  is not unique, however, imposing stronger condition that  $\|f^c - f\|_\infty \leq c/2$  and for any  $s \in (a; b]$

$$(1.3) \quad TV(f^c, [a; s]) = TV^c(f, [a; s]),$$

we will obtain that the function  $f^c$  exists and is uniquely determined for any  $c \leq \sup_{s, u \in [a; b]} |f(s) - f(u)|$  (cf. Corollary 3.8).

**Remark 1.2.** The natural question appears if the truncated variation is attainable lower bound for  $\inf_{g \in B(f, c/2)} TV(g, [a; b])$  for functions with values

in other metric spaces, but the answer to this problem is not known to the author.

Since for  $E = \mathbb{R}$  with  $\rho(x, y) = |x - y|$  the total variation depends only on the increments of the function, in this case a more natural problem would be the following. For a càdlàg function  $f : [a; b] \rightarrow \mathbb{R}$  and  $c > 0$  find

$$\inf \{TV(f + h, [a; b]) : \|h\|_{osc} \leq c\},$$

where for  $h : [a; b] \rightarrow \mathbb{R}$ ,  $\|h\|_{osc} := \sup_{s, u \in [a; b]} |h(s) - h(u)|$ . Note that  $\|\cdot\|_{osc}$  is a norm on the classes of bounded functions which differ by a constant.

Solution to this problem is the same as the solution to the preceding problem, i.e.

$$(1.4) \quad \inf \{TV(f + h, [a; b]) : \|h\|_{osc} \leq c\} = TV^c(f, [a; b])$$

and one of the optimal representatives of the class of functions for which equality (1.4) is attained is  $h^c = f^c - f$ . To this class also belongs some  $h^{0,c}$ , such that  $h^{0,c}(a) = 0$ . We will prove that  $f^{0,c} = f + h^{0,c} - f(a)$  is a càdlàg function with possible jumps only in the points where the function  $f$  has jumps and that it may be represented in the following form

$$(1.5) \quad f^{0,c}(s) = UTV^c(f; [a; s]) - DTV^c(f; [a; s]),$$

where

$$(1.6) \quad UTV^c(f, [a; s]) := \sup_n \sup_{a \leq t_0 < t_1 < \dots < t_n \leq s} \sum_{i=1}^n \max\{f(t_i) - f(t_{i-1}) - c, 0\},$$

$$(1.7) \quad DTV^c(f, [a; s]) := \sup_n \sup_{a \leq t_0 < t_1 < \dots < t_n \leq s} \sum_{i=1}^n \max\{f(t_{i-1}) - f(t_i) - c, 0\}.$$

The functionals  $UTV^c(f, [a; s])$  and  $DTV^c(f, [a; s])$  are non-decreasing functions of  $s$  and are called upward and downward truncated variations of the function  $f$  of order  $c$  on the interval  $[a; s]$  respectively. They were first introduced in [L2] with a bit different formulas, equivalent with (1.6) and (1.7).

Finally, for  $s \in (a; b]$  we will show the following equality

$$(1.8) \quad TV(f^{0,c}, [a; s]) = TV^c(f, [a; s]) = UTV^c(f, [a; s]) + DTV^c(f, [a; s]).$$

The equalities (1.5) and (1.8) give the Hahn-Jordan decomposition of the finite signed measure, induced by the function  $f^{0,c}$  (or by the function  $f^c$ ).

This measure assigns to any interval  $(a_1, b_1] \subset (a; b]$  the number

$$\mu(a_1, b_1] = f^{0,c}(b_1) - f^{0,c}(a_1)$$

and we have

$$\mu(a_1, b_1] = \mu_+(a_1, b_1] - \mu_-(a_1, b_1],$$

where

$$\begin{aligned}\mu_+(a_1, b_1] &= UTV^c(f, [a; b_1]) - UTV^c(f, [a; a_1]), \\ \mu_-(a_1, b_1] &= DTV^c(f, [a; b_1]) - DTV^c(f, [a; a_1]).\end{aligned}$$

However, since  $c > 0$  is arbitrary, the equalities (1.5) and (1.8) also may be viewed as the generalisation of the Hahn-Jordan decomposition for any real càdlàg function  $f$ .

**Remark 1.3.** Since we deal with càdlàg functions, a more natural setting of the first problem would be the investigation of

$$\inf \{TV(g, [a; b]) : g \text{ - càdlàg}, d_D(g, f) \leq c/2\},$$

where  $d_D$  denotes the Skorohod metric (cf. [B, Chapt. 3]). However, the total variation does not depend on the (continuous and strictly increasing) transformations of the argument and for  $E = \mathbb{R}$  with  $\rho(x, y) = |x - y|$  the function  $f^c$  minimizing  $TV(g, [a; b])$  appears to be a càdlàg one, hence solutions of both problems coincide in this case.

Let us comment on the organisation of the paper. In the next section we deal with functions attaining values in metric spaces and prove Fact 2.1. In the third section we deal with real càdlàg functions - introduce some necessary definitions and notation and present the construction of the functions  $f^c$  and  $f^{0,c}$  of the first and the second problem. In the fourth section we establish the connection between  $f^{0,c}$  and truncated variation, upward truncated variation and downward truncated variation. In the last section we summarize some other general properties of (upward, downward) truncated variation, e.g. we will show that for any real càdlàg function  $f$ ,  $TV^c(f, [a; b])$  is a continuous, convex and decreasing function of the parameter  $c > 0$ .

## 2. TRUNCATED VARIATION OF FUNCTIONS ATTAINING VALUES IN METRIC SPACES

In this section we consider families of functions  $f : [a; b] \rightarrow E$ , with finite truncated variation for any  $c > 0$ , even if their total variation appears to be infinite. The natural example of a family of such functions are functions which are uniform limits of sequences of step functions. When the space  $E$  is complete, then this family coincides with the family of functions with left

and right limits (cf. [D, Theorem 7.6.1]). When  $E$  is a Banach space, such functions are called *regulated functions* (see [D, Sect. VII.6]).

However, when we restrict ourselves to càdlàg functions, then we do not need any additional assumption on the space  $E$  and we have the following

**Fact 2.1.** *Whenever  $f$  is càdlàg and  $c > 0$  then it may be uniformly approximated with accuracy  $c/2$  by a step càdlàg function and hence  $TV^c(f, [a; b])$  is finite.*

*Proof.* Let us define infinite sequence of points  $a = t_0 < t_1 < \dots \leq +\infty$  in the following way: for  $i = 1, 2, \dots$

$$t_i = \begin{cases} \inf \{t \in (t_{i-1}; b] : \rho(f(t), f(t_{i-1})) > c/2\} & \text{if } t_{i-1} < b; \\ +\infty & \text{otherwise.} \end{cases}$$

Since  $f$  is right continuous, then for  $t_{i-1} \leq b$  we have  $t_i > t_{i-1}$ . Thus there must exists some  $i_0 < +\infty$  such that  $t_i = +\infty$  for  $i \geq i_0$  (recall that  $\inf = +\infty$ ). Otherwise we would obtain an infinite sequence  $a = t_0 < t_1 < \dots < b$  such that  $\rho(f(t_{i-1}), f(t_i)) > c/2$  for  $i = 1, 2, \dots$ ; but this contradicts the fact that for any sequence  $a = t_0 < t_1 < \dots < b$ , the sequence  $(f(t_i))_{i=0}^\infty$  has a limit, due to the càdlàg property of the function  $f$ . Now define the function  $g : [a; b] \rightarrow E$  as a step function,

$$g(t) = f(t_i) \text{ if } t \in [t_i; t_{i+1}).$$

(This definition is correct, since  $[a; b] \subset \bigcup_{i \geq 0} [t_i; t_{i+1})$ .) We notice that  $g \in B(f, c/2)$  and  $TV(g, [a; b]) < +\infty$ , hence

$$TV^c(f, [a; b]) \leq TV(g, [a; b]) < +\infty.$$

□

**Remark 2.2.** From (1.2) we may derive some upper bound for

$$\inf_{g \in B(f, c/2)} TV(g, [a; b])$$

when  $f$  is càdlàg and  $E = \mathbb{R}^N$  with  $\rho$  induced by the  $L^1$  norm. Namely, for  $f(t) = (f_1(t), \dots, f_N(t)) \in \mathbb{R}^N$ ,  $\|f(t)\|_1 := |f_1(t)| + \dots + |f_N(t)|$  and  $\rho(f(t), g(t)) := \|f(t) - g(t)\|_1$ , we have

$$\begin{aligned} \inf_{g \in B(f, c/2)} TV(g, [a; b]) &\leq \inf_{c_1, \dots, c_N > 0, c_1 + \dots + c_N = c} \sum_{i=1}^N \inf_{g_i \in B(f_i, c_i/2)} TV(g_i, [a; b]) \\ &= \inf_{c_1, \dots, c_N > 0, c_1 + \dots + c_N = c} \sum_{i=1}^N TV^{c_i}(f_i, [a; b]). \end{aligned}$$

### 3. SOLUTION OF THE FIRST AND THE SECOND PROBLEM FOR REAL CÀDLÀG FUNCTIONS

**3.1. Definitions and notation.** In this subsection we introduce definitions and notation which will be used throughout the whole paper.

Let  $f : [a; b] \rightarrow \mathbb{R}$  be a càdlàg function. For  $c > 0$  we define two stopping times

$$T_D^c f = \inf \left\{ s \geq a : \sup_{t \in [a; s]} f(t) - f(s) \geq c \right\},$$

$$T_U^c f = \inf \left\{ s \geq a : f(s) - \inf_{t \in [a; s]} f(t) \geq c \right\}.$$

Assume that  $T_D^c f \geq T_U^c f$  i.e. the first upward jump of function  $f$  of size  $c$  appears before the first downward jump of the same size  $c$  or both times are infinite (there is no upward neither downward jump of size  $c$ ). Note that in the case  $T_D^c f < T_U^c f$  we may simply consider function  $-f$ . Now we define sequences  $(T_{U,k}^c)_{k=0}^\infty, (T_{D,k}^c)_{k=-1}^\infty$ , in the following way:  $T_{D,-1}^c = a$ ,  $T_{U,0}^c = T_U^c f$  and for  $k = 0, 1, 2, \dots$

$$T_{D,k}^c = \begin{cases} \inf \left\{ s \in [T_{U,k}^c; b] : \sup_{t \in [T_{U,k}^c; s]} f(t) - f(s) \geq c \right\} & \text{if } T_{U,k}^c < b, \\ \infty & \text{if } T_{U,k}^c \geq b, \end{cases}$$

$$T_{U,k+1}^c = \begin{cases} \inf \left\{ s \in [T_{D,k}^c; b] : f(s) - \inf_{t \in [T_{D,k}^c; s]} f(t) \geq c \right\} & \text{if } T_{D,k}^c < b, \\ \infty & \text{if } T_{D,k}^c \geq b. \end{cases}$$

**Remark 3.1.** Note that there exists such  $K < \infty$  that  $T_{U,K}^c = \infty$  or  $T_{D,K}^c = \infty$ . Otherwise we would obtain two infinite sequences  $(s_k)_{k=1}^\infty, (S_k)_{k=1}^\infty$  such that  $a \leq s_1 < S_1 < s_2 < S_2 < \dots \leq b$  and  $f(S_k) - f(s_k) \geq c/2$ . But this is a contradiction, since  $f$  is a càdlàg function and  $(f(s_k))_{k=1}^\infty, (f(S_k))_{k=1}^\infty$  have a common limit.

Now let us define two sequences of non-decreasing functions  $m_k^c : [T_{D,k-1}^c; T_{U,k}^c) \cap [a; b] \rightarrow \mathbb{R}$  and  $M_k^c : [T_{U,k}^c; T_{D,k}^c) \cap [a; b] \rightarrow \mathbb{R}$  with the formulas

$$m_k^c(s) = \inf_{t \in [T_{D,k-1}^c; s]} f(t), \quad M_k^c(s) = \sup_{t \in [T_{U,k}^c; s]} f(t).$$

Next we define two finite sequences of real numbers  $(m_k^c)$  and  $(M_k^c)$ , for such  $k$  that  $T_{D,k-1}^c < \infty$  and  $T_{U,k}^c < \infty$  respectively, with the formulas

$$m_k^c = m_k^c(T_{U,k}^c -) = \inf_{t \in [T_{D,k-1}^c; T_{U,k}^c) \cap [a; b]} f(t),$$

$$M_k^c = M_k^c(T_{D,k}^c -) = \sup_{t \in [T_{U,k}^c; T_{D,k}^c) \cap [a; b]} f(t).$$

**3.2. Solution of the first problem.** In this subsection we will solve the following problem: *what is the smallest possible (or infimum of) total variation of functions from the ball  $\{g : \|f - g\|_\infty \leq c/2\}$ ?*

In order to solve this problem we start with results concerning càdlàg functions. We apply the definitions of the previous subsection to the function  $f$  and assume that  $T_D^c f \geq T_U^c f$ . Define the function  $f^c : [a; b] \rightarrow \mathbb{R}$  with the formulas

$$f^c(s) = \begin{cases} m_0^c + c/2 & \text{if } s \in [a; T_{U,0}^c); \\ M_k^c(s) - c/2 & \text{if } s \in [T_{U,k}^c; T_{D,k}^c), k = 0, 1, 2, \dots; \\ m_{k+1}^c(s) + c/2 & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c), k = 0, 1, 2, \dots \end{cases}$$

**Remark 3.2.** Note that due to Remark 3.1,  $b$  belongs to one of the intervals  $[T_{U,k}^c; T_{D,k}^c)$  or  $[T_{D,k}^c; T_{U,k+1}^c)$  for some  $k = 0, 1, 2, \dots$  and the function  $f^c$  is defined for every  $s \in [a; b]$ .

**Remark 3.3.** One may think about the function  $f^c$  as of the most "lazy" function possible, which changes its value only if it is necessary for the relation  $\|f - f^c\|_\infty \leq c/2$  to hold. The choice of its starting value will become clear in the sequel.

**Remark 3.4.** In the case  $T_D^c f < T_U^c f$  we may apply the definitions of the previous subsection to the function  $-f$  and simply define  $f^c = -(-f)^c$ . Thus we will assume that the mapping  $f \mapsto f^c$  is defined for any càdlàg function. Similarly, in all the proofs of this section we will assume  $T_D^c f \leq T_U^c f$ , but all results of this section (i.e. Lemma 3.5, Theorem 3.6, Corollary 3.8, Lemma 3.9, Theorem 3.10, Corollary 3.11 and Theorem 4.1) apply to any càdlàg function  $f$ . Obvious modifications are only necessary in the definition of the stopping times  $T_{U,k}^c$  and  $T_{D,k}^c$  and then the functions  $f_U^c$  and  $f_D^c$  of Theorem 3.6.

We have the following

**Lemma 3.5.** *The function  $f^c$  uniformly approximates the function  $f$  with accuracy  $c/2$  and has finite total variation. Moreover  $f^c$  is a càdlàg function and every point of the discontinuity of  $f^c$  is also a point of discontinuity of the function  $f$ .*

*Proof.* Let us fix  $s \in [a; b]$ . We have three possibilities.

- $s \in [a; T_{U,0}^c)$ . In this case, since  $a \leq s < T_U^c f \leq T_D^c f$ ,

$$f(s) - f^c(s) = f(s) - \inf_{t \in [a; T_{U,0}^c)} f(t) - c/2 \in [-c/2; c/2].$$

- $s \in [T_{U,k}^c; T_{D,k}^c)$ , for some  $k = 0, 1, 2, \dots$ . In this case  $M_k^c(s) - f(s)$  belongs to the interval  $[0; c)$ , hence

$$f(s) - f^c(s) = f(s) - M_k^c(s) + c/2 \in (-c/2; c/2].$$

- $s \in [T_{D,k}^c; T_{U,k+1}^c)$  for some  $k = 0, 1, 2, \dots$ . In this case  $f(s) - m_{k+1}^c(s)$  belongs to the interval  $[0, c)$ , hence

$$f(s) - f^c(s) = f(s) - m_{k+1}^c(s) - c/2 \in [-c/2; c/2).$$

The function  $f^c$  has finite total variation since it is non-decreasing on the intervals  $[T_{U,k}^c; T_{D,k}^c)$ ,  $k = 0, 1, 2, \dots$  and non-increasing on the intervals  $[T_{D,k}^c; T_{U,k+1}^c)$ ,  $k = 0, 1, 2, \dots$ , and it has finite number of jumps between these intervals.

For a similar reason, the function  $f^c$  has left and right limits. To see that it is right-continuous, let us fix  $s \in [a; b]$  and notice that by definition of  $f^c$ , for  $t \in (s; b]$  sufficiently close to  $s$ ,

$$f^c(t) = \inf_{u \in [s; t]} f^c(u) \text{ or } f^c(t) = \sup_{u \in [s; t]} f^c(u),$$

and the assertion follows from the right-continuity of the function  $f$ .

A similar argument may be applied to prove that  $f^c$  is continuous in every point of continuity of  $f$  except the points  $T_{U,0}^c, T_{D,0}^c, T_{U,1}^c, T_{D,1}^c, \dots$ ; but if  $s = T_{D,i}^c$  and  $f$  is continuous at the point  $s$  then it means that  $f(T_{U,i}^c-) = f(T_{U,i}^c) = \inf_{t \in [T_{D,i-1}^c; T_{U,i}^c)} f(t) + c$  and

$$f^c(T_{U,i}^c-) = \inf_{t \in [T_{D,i-1}^c; T_{U,i}^c)} f(t) + c/2 = f(T_{U,i}^c) - c/2 = f^c(T_{U,i}^c).$$

A similar argument applies when  $s = T_{D,i}^c$ .

□

Since  $f^c$  is of finite total variation, we know that there exist such two non-decreasing functions  $f_U^c$  and  $f_D^c : [a; b] \rightarrow [0; +\infty)$  that  $f^c(t) = f^c(a) + f_U^c(t) - f_D^c(t)$ .

Let us examine the signs of the jumps of the function  $f^c$  between intervals  $[T_{U,k}^c; T_{D,k}^c)$  and  $[T_{D,k}^c; T_{U,k+1}^c)$ . Due to càdlàg property we have

$$\begin{aligned} f^c(T_{U,k}^c) - f^c(T_{U,k}^c-) &= f^c(T_{U,k}^c) - m_k^c - c \\ &= f(T_{U,k}^c) - \inf_{t \in [T_{D,k-1}^c; T_{U,k}^c)} f(t) - c \geq 0, \\ f^c(T_{D,k}^c) - f^c(T_{D,k}^c-) &= f^c(T_{D,k}^c) - M_k^c + 2c \\ &= - \left\{ \sup_{t \in [T_{U,k}^c; T_{D,k}^c)} f(t) - f(T_{D,k}^c) \right\} + c \leq 0. \end{aligned}$$



Hence we may set  $f_U^c(s) = f_D^c(s) = 0$  for  $s \in [a; T_{U,0}^c)$ ,

$$f_U^c(s) = \begin{cases} \sum_{i=0}^{k-1} \{M_i^c - m_i^c - c\} + M_k^c(s) - m_k^c - c & \text{if } s \in [T_{U,k}^c; T_{D,k}^c); \\ \sum_{i=0}^k \{M_i^c - m_i^c - c\} & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c) \end{cases}$$

and

$$f_D^c(s) = \begin{cases} \sum_{i=0}^{k-1} \{M_i^c - m_{i+1}^c - c\} & \text{if } s \in [T_{U,k}^c; T_{D,k}^c); \\ \sum_{i=0}^{k-1} \{M_i^c - m_{i+1}^c - c\} + M_k^c - m_{k+1}^c(s) - c & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c). \end{cases}$$

Now we will prove the following

**Theorem 3.6.** *If  $g : [a; b] \rightarrow \mathbb{R}$  uniformly approximates  $f$  with accuracy  $c/2$ , has finite total variation and  $g_U, g_D : [a; b] \rightarrow [0; +\infty)$  are such two non-decreasing functions that  $g(t) = g(a) + g_U(t) - g_D(t)$ ,  $t \in [a; b]$ , then for any  $s \in [a; b]$*

$$(3.1) \quad g_U(s) \geq f_U^c(s) \text{ and } g_D(s) \geq f_D^c(s).$$

*Proof.* Again, we consider three cases.

- $s \in [a; T_{U,0}^c)$ . In this case  $g_U(s) \geq 0 = f_U^c(s)$  as well as  $g_D(s) \geq 0 = f_D^c(s)$
- $s \in [T_{U,k}^c; T_{D,k}^c)$ , for some  $k = 0, 1, 2, \dots$ . In this case, from the fact that  $g$  uniformly approximates  $f$  with accuracy  $c/2$  and from the fact that  $g_U, g_D$  are non-decreasing, for  $i = 0, 1, 2, \dots, k-1$  we get

$$\begin{aligned} & \sup_{s_i \in [T_{U,i}^c; T_{D,i}^c)} g_U(s_i) - \inf_{s_i \in [T_{D,i-1}^c; T_{U,i}^c)} g_U(s_i) \\ & \geq \sup_{s_i \in [T_{U,i}^c; T_{D,i}^c)} (g_U - g_D)(s_i) - \inf_{s_i \in [T_{D,i-1}^c; T_{U,i}^c)} (g_U - g_D)(s_i) \\ & = \sup_{s_i \in [T_{U,i}^c; T_{D,i}^c)} g(s_i) - \inf_{s_i \in [T_{D,i-1}^c; T_{U,i}^c)} g(s_i) \\ & \geq \sup_{s_i \in [T_{U,i}^c; T_{D,i}^c)} \{f(s_i) - c/2\} - \inf_{s_i \in [T_{D,i-1}^c; T_{U,i}^c)} \{f(s_i) + c/2\} \\ & = M_i^c - m_i^c - c. \end{aligned}$$

Similarly

$$\begin{aligned}
& g_U(s) - \inf_{s_k \in [T_{D,k-1}^c; T_{U,k}^c)} g_U(s_k) \\
&= \sup_{t \in [T_{U,k}^c; s]} g_U(t) - \inf_{s_k \in [T_{D,k-1}^c; T_{U,k}^c)} g_U(s_k) \\
&\geq \sup_{t \in [T_{U,k}^c; s]} (g_U - g_D)(t) - \inf_{s_k \in [T_{D,k-1}^c; T_{U,k}^c)} (g_U - g_D)(s_k) \\
&= \sup_{t \in [T_{U,k}^c; s]} g(t) - \inf_{s_k \in [T_{D,k-1}^c; T_{U,k}^c)} g(s_k) \\
&\geq \sup_{t \in [T_{U,k}^c; s]} \{f(t) - c/2\} - \inf_{s_k \in [T_{D,k-1}^c; T_{U,k}^c)} \{f(s_k) + c/2\} \\
&= M_k^c(s) - m_k^c - c.
\end{aligned}$$

Summing up the above inequalities and using monotonicity of  $g_U$  we finally get

$$g_U(s) \geq \sum_{i=0}^{k-1} \{M_i^c - m_i^c - c\} + M_k^c(s) - m_k^c - c = f_U^c(s).$$

The proof of the corresponding inequality for  $g_D$  follows similarly and we get

$$g_D(s) \geq \sum_{i=0}^{k-1} \{M_i^c - m_{i+1}^c - c\} = f_D^c(s).$$

- $s \in [T_{D,k}^c; T_{U,k+1}^c)$  The proof follows similarly as in the previous case.

□

From Theorem 3.6 we immediately get that the decomposition

$$(3.2) \quad f^c(s) = f^c(a) + f_U^c(s) - f_D^c(s)$$

is minimal (cf. [RY], page 5) thus the total variation of the function  $f^c$  on the interval  $[a; s]$  equals  $f_U^c(s) + f_D^c(s)$ .

**Remark 3.7.** From Lemma 3.5 and the minimality of the decomposition (3.2) it follows that  $f_U^c$  and  $f_D^c$  are also càdlàg functions and that every point of their discontinuity is also a point of discontinuity of the function  $f$ . Moreover, due to the minimality of the variation of the function  $f^c$ , any jump of  $f^c$  is no greater than the jump of the function  $f$ .

We also have

**Corollary 3.8.** *The function  $f^c$  is optimal i.e. if  $g : [a; b] \rightarrow \mathbb{R}$  is such that  $\|f - g\|_\infty \leq c/2$  and has finite total variation, then for every  $s \in [a; b]$*

$$TV(g, [a; s]) \geq TV(f^c, [a; s]).$$

Moreover, it is unique in such a sense that if for every  $s \in [a; b]$  the opposite inequality holds

$$TV(g, [a; s]) \leq TV(f^c, [a; s])$$

and  $c \leq \sup_{s, u \in [a; b]} |f(s) - f(u)|$  then  $g = f^c$ .

*Proof.* Let  $g_U, g_D : [a; b] \rightarrow [0; +\infty)$  be two non-decreasing functions such that for  $s \in [a; b]$ ,  $g(s) = g(a) + g_U(s) - g_D(s)$  and  $TV(g, [a; s]) = g_U(s) + g_D(s)$ .

The first assertion follows directly from Theorem 3.6 and the fact that  $TV(g, [a; s]) = g_U(s) + g_D(s)$ .

The opposite inequality holds for every  $s \in [a; b]$  iff  $g_U(s) = f_U^c(s)$  and  $g_D(s) = f_D^c(s)$ . Thus in such a case we get  $g(s) - f^c(s) = g(a) - f^c(a)$  and we have

$$\begin{aligned} c/2 &\geq \inf_{s \in [a; T_{U,0}^c]} \{g(s) - f(s)\} = \inf_{s \in [a; T_{U,0}^c]} \{g(a) - f^c(a) + f^c(s) - f(s)\} \\ (3.3) &= g(a) - f^c(a) + c/2 \end{aligned}$$

(notice that  $T_{U,0}^c \leq b$  since  $c \leq \sup_{s, u \in [a; b]} |f(s) - f(u)|$  and  $T_{U,0}^c \leq T_{D,0}^c$ ). On the other hand we have

$$\begin{aligned} -c/2 &\leq g(T_{U,0}^c) - f(T_{U,0}^c) = g(a) - f^c(a) + f^c(T_{U,0}^c) - f(T_{U,0}^c) \\ (3.4) &= g(a) - f^c(a) - c/2. \end{aligned}$$

From (3.3) and (3.4) we get  $g(a) = f^c(a)$ . This together with the equalities  $g_U(s) = f_U^c(s)$  and  $g_D(s) = f_D^c(s)$  gives  $g = f^c$ .  $\square$

The formula obtained for the smallest possible total variation of a function from the ball  $\{g : \|f - g\| \leq c/2\}$  reads as

$$f_U^c(b) + f_D^c(b)$$

and does not resemble formula (1.2). In subsection 4 we will show that these formulas coincide.

**3.3. Solution of the second problem.** In this subsection we will solve the following problem: *for a càdlàg function  $f : [a; b] \rightarrow \mathbb{R}$  and  $c > 0$  find*

$$\inf \{TV(f + h, [a; b]) : \|h\|_{osc} \leq c\},$$

where  $h : [a; b] \rightarrow \mathbb{R}$ ,  $\|h\|_{osc} := \sup_{s, u \in [a; b]} |h(s) - h(u)|$ .

We will show that

$$\inf \{TV(f + h, [a; b]) : \|h\|_{osc} \leq c\} = f_U^c(b) + f_D^c(b),$$

where  $f_U^c$  and  $f_D^c$  were defined in the previous subsection. In order to do it let us simply define

$$f^{0,c} = f_U^c - f_D^c.$$

We have

**Lemma 3.9.** *The increments of the function  $f^{0,c}$  uniformly approximate the increments of the function  $f$  with accuracy  $c$  and the function  $f^{0,c}$  has finite total variation.*

*Proof.* Since the difference  $f^c - f^{0,c}$  is constant, the first and the second assertion follows immediately from Lemma 3.5 and from simple calculation that for any  $s, u \in [a; b]$ ,

$$\begin{aligned} & \{f^{0,c}(s) - f^{0,c}(u)\} - \{f(s) - f(u)\} \\ &= \{f^c(s) - f(s)\} - \{f^c(u) - f(u)\} \in [-c; c]. \end{aligned}$$

□

Now we will prove the analog of Theorem 3.6.

**Theorem 3.10.** *If the increments of the function  $g : [a; b] \rightarrow \mathbb{R}$  uniformly approximate the increments of the function  $f$  with accuracy  $c$ ,  $g$  has finite total variation and  $g_U, g_D : [a; b] \rightarrow [0; +\infty)$  are such two non-decreasing functions that  $g(t) = g(a) + g_U(t) - g_D(t)$ ,  $t \in [a; b]$ , then for any  $s \in [a; b]$*

$$g_U(s) \geq f_U^c(s) \text{ and } g_D(s) \geq f_D^c(s).$$

*Proof.* It is enough to see that for  $h = g - f$ ,  $\|h\|_{osc} \leq c$ , thus for

$$\alpha = -\frac{1}{2} \left\{ \inf_{s \in [a; b]} h(s) + \sup_{s \in [a; b]} h(s) \right\},$$

$\|\alpha + h\|_\infty \leq \frac{1}{2}c$ , and the function  $g_\alpha = \alpha + g$  belongs to the ball  $\{g : \|f - g\|_\infty \leq \frac{1}{2}c\}$ . Application of Theorem 3.6 to the function  $g_\alpha$  concludes the proof.

□

Since the decomposition  $f^{0,c}(s) = f_U^c(s) - f_D^c(s)$  is minimal and  $f^{0,c}(a) = 0$  we immediately obtain

**Corollary 3.11.** *The function  $f^{0,c}$  is optimal i.e. if  $g : [a; b] \rightarrow \mathbb{R}$  is such that*

$$\sup_{a \leq u < s \leq b} |\{g(s) - g(u)\} - \{f(s) - f(u)\}| \leq c$$

*and  $g$  has finite total variation, then for every  $s \in [a; b]$*

$$TV(g, [a; s]) \geq TV(f^{0,c}, [a; s]).$$

Moreover, it is unique in such a sense that if  $g(a) = 0$  and for every  $s \in [a; b]$  the opposite inequality holds

$$TV(g, [a; s]) \leq TV(f^{0,c}, [a; s]),$$

then  $g = f^{0,c}$ .

From Corollary 3.11 it immediately follows that

$$\inf \{TV(f + h, [a; b]) : \|h\|_{osc} \leq c\} = f_U^c(b) + f_D^c(b).$$

Indeed, for any  $h$  such that  $\|h\|_{osc} \leq c$  we put  $g = f + h$  and if  $g$  has finite total variation then it satisfies the assumptions of Corollary 3.11 and we get

$$TV(g, [a; b]) \geq TV(f^{0,c}, [a; b]) = f_U^c(b) + f_D^c(b).$$

#### 4. RELATION OF THE SOLUTIONS OF THE FIRST AND THE SECOND PROBLEM WITH TRUNCATED VARIATION, UPWARD TRUNCATED VARIATION AND DOWNWARD TRUNCATED VARIATION

In order to prove (1.2), (1.4) and (1.8), where  $UTV^c(f, [a; s])$  and  $DTV^c(f, [a; s])$  are defined by (1.6) and (1.7) respectively, it is enough to prove

**Theorem 4.1.** *For a given càdlàg function  $f : [a; b] \rightarrow \mathbb{R}$  and for any  $s \in (a; b]$  the following equalities hold*

$$(4.1) \quad UTV^c(f, [a; s]) = f_U^c(s),$$

$$(4.2) \quad DTV^c(f, [a; s]) = f_D^c(s),$$

$$(4.3) \quad TV^c(f, [a; s]) = f_U^c(s) + f_D^c(s).$$

*Proof.* Examining (with obvious modifications) the proof of Lemma 3 from [L2], we see that it may be applied to the càdlàg (but not necessarily continuous) function  $f$  and we obtain

$$(4.4) \quad UTV^c(f, [a; s]) = \sup_{a \leq t < u \leq (T_D^c f) \wedge s} (f(u) - f(t) - c)_+ + UTV^c(f, [(T_D^c f) \wedge s; s]).$$

Now, from the assumption  $T_D^c f \geq T_U^c f$  we get  $T_D^c f = T_{D,0}^c$  and we have that

$$\sup_{a \leq t < u \leq (T_D^c f) \wedge s} (f(u) - f(t) - c)_+ = \begin{cases} 0 & \text{if } s \in [a; T_{U,0}^c); \\ M_0^c(s) - m_0^c - c & \text{if } s \in [T_{U,0}^c; T_{D,0}^c); \\ M_0^c - m_0^c - c & \text{if } s \geq T_{D,0}^c. \end{cases}$$

Iterating the equality (4.4) we obtain

$$\begin{aligned} UTV^c(f, [a; s]) &= \begin{cases} 0 & \text{if } s \in [a; T_{U,0}^c); \\ \sum_{i=0}^{k-1} (M_i^c - m_i^c - c) + M_k^c(s) - m_k^c - c & \text{if } s \in [T_{U,k}^c; T_{D,k}^c); \\ \sum_{i=0}^k (M_i^c - m_i^c - c) & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c) \end{cases} \\ &= f_U^c(s). \end{aligned}$$

**Remark 4.2.** Iterating (4.4) we obtain a bit different equality than  $UTV^c(f, [a; s]) = f_U^c$ , but equivalent with it. To see this let us define the following sequence of stopping times.  $\tilde{T}_{D,-1}^c = a$ , and for  $k = 0, 1, 2, \dots$

$$\tilde{T}_{D,k+1}^c = \inf \left\{ s > \tilde{T}_{D,k}^c : \sup_{t \in [\tilde{T}_{D,k}^c; s]} f(t) - f(s) = c \right\}.$$

Let us fix  $s_0 \in [a; b]$  and define  $k_0 = \max \left\{ k : \tilde{T}_{D,k}^c \leq s_0 \right\}$ . Iterating (4.4) we obtain the following equality

$$UTV^c(f, [a; s_0]) = \sum_{k=1}^{k_0-1} \sup_{\tilde{T}_{D,k}^c \leq s < u \leq \tilde{T}_{D,k+1}^c} (f(u) - f(s) - c)_+ + UTV^c\left(f, \left[\tilde{T}_{D,k_0}^c; s_0\right]\right)$$

which looks different from  $f_U^c(s_0)$ . But it is easy to notice that for all  $k \geq 1$  such that  $\tilde{T}_{D,k+1}^c < T_{U,1}^c$ , the summand  $\sup_{\tilde{T}_{D,k}^c \leq s < u \leq \tilde{T}_{D,k+1}^c} (f(u) - f(s) - c)_+$  is equal zero. Thus in fact both quantities coincide.

Identically we prove that  $DTV^c(f)[a; s] = f_D^c(s)$ .

Now, in order to prove the equality (4.3) simply notice that  $TV^c(f, [a; s]) \geq 0$  and if  $s \in [T_{U,k}^c; T_{D,k}^c)$

$$\begin{aligned} TV^c(f, [a; s]) &\geq \sum_{i=0}^{k-1} (M_i^c - m_i^c - c) + \sum_{i=0}^{k-1} (M_i^c - m_{i+1}^c - c) + M_k^c(s) - m_k^c - c \\ &= f_U^c(s) + f_D^c(s). \end{aligned}$$

Analogously, if  $s \in [T_{D,k}^c; T_{U,k+1}^c)$

$$\begin{aligned} TV^c(f, [a; s]) &\geq \sum_{i=0}^{k-1} (M_i^c - m_i^c - c) + \sum_{i=0}^{k-1} (M_i^c - m_{i+1}^c - c) + M_k^c - m_{k+1}^c(s) - c \\ &= f_U^c(s) + f_D^c(s). \end{aligned}$$

Hence for all  $s \in [a; b]$

$$TV^c(f, [a; s]) \geq f_U^c(s) + f_D^c(s).$$

So

$$TV^c(f, [a; s]) \geq UTV^c(f, [a; s]) + DTV^c(f, [a; s]).$$

Since the opposite inequality is obvious, we finally get (4.3).  $\square$

Now we see that by Corollary 3.8 and Corollary 3.11 functions  $h^c = f^c - f$  and  $h^{0,c} = f(a) + f^{0,c} - f = f(a) + UTV^c(f, [a; \cdot]) - DTV^c(f, [a; \cdot]) - f$  are optimal and such that for any  $s \in (a; b]$

$$\begin{aligned} \inf \{TV(f + h, [a; s]) : \|h\|_\infty \leq c/2\} &= TV(f + h^c, [a; s]) \\ &= TV^c(f, [a; s]), \end{aligned}$$

$$\begin{aligned} \inf \{TV(f+h, [a; s]) : \|h\|_{osc} \leq c\} &= TV(f+h^{0,c}, [a; s]) \\ &= TV^c(f, [a; s]). \end{aligned}$$

Moreover, by Remark 3.7,  $h^c$  and  $h^{0,c}$  are also càdlàg functions and every point of their discontinuity is also a point of discontinuity of the function  $f$ .

## 5. FURTHER PROPERTIES OF TRUNCATED VARIATION, UPWARD TRUNCATED VARIATION AND DOWNWARD TRUNCATED VARIATION

In this section we summarize basic properties of the defined functionals. We start with

**5.1. Algebraic properties.** For any  $c > 0$  we have

$$(5.1) \quad DTV^c(f, [a; b]) = UTV^c(-f, [a; b]),$$

$$(5.2) \quad TV^c(f, [a; b]) = UTV^c(f, [a; b]) + DTV^c(f, [a; b]).$$

Property (5.1) follows simply from the definitions (1.6) and (1.7). Property (5.2) is the consequence of Theorem 4.1.

**5.2. Properties of  $UTV^c(f, [a; b])$ ,  $DTV^c(f, [a; b])$  and  $TV^c(f, [a; b])$  as the functions of the parameter  $c$ .** We have the following

**Fact 5.1.** *For any càdlàg function  $f$  the functions  $(0; \infty) \ni c \mapsto UTV^c(f, [a; b]) \in [0; +\infty)$ ,  $(0; \infty) \ni c \mapsto DTV^c(f, [a; b]) \in [0; +\infty)$  and  $(0; \infty) \ni c \mapsto TV^c(f, [a; b]) \in [0; +\infty)$  are nonincreasing, continuous, convex functions of the parameter  $c$ . Moreover,  $\lim_{c \downarrow 0} TV^c(f, [a; b]) = TV(f, [a; b])$  and for any  $c \geq \|f\|_{osc}$ ,  $TV^c(f, [a; b]) = 0$ .*

*Proof.* The finiteness of  $TV$ ,  $UTV$  and  $DTV$  follows from Lemma 3.5 and Theorem 4.1. Monotonicity is obvious.

We start with the proof of the convexity. Let us fix  $c, \varepsilon > 0$  and consider such a partition  $a \leq t_0 < t_1 < \dots < t_n \leq b$  of the interval  $[a; b]$  that

$$UTV^c(f, [a; b]) \leq \sum_{i=0}^{n-1} \max \{f(t_{i+1}) - f(t_i) - c, 0\} + \varepsilon.$$

Taking  $\alpha \in [0; 1]$  and  $c_1, c_2 > 0$  such that  $c = \alpha c_1 + (1 - \alpha) c_2$  we have the inequality

$$\begin{aligned} &\max \{f(t_{i+1}) - f(t_i) - \alpha c_1 - (1 - \alpha) c_2, 0\} \\ &= \max \{\alpha (f(t_{i+1}) - f(t_i) - c_1) + (1 - \alpha) (f(t_{i+1}) - f(t_i) - c_2), 0\} \\ &\leq \alpha \max \{f(t_{i+1}) - f(t_i) - c_1, 0\} + (1 - \alpha) \max \{f(t_{i+1}) - f(t_i) - c_2, 0\}. \end{aligned}$$

Now

$$\begin{aligned}
UTV^c(f, [a; b]) &\leq \sum_{i=0}^{n-1} \max \{f(t_{i+1}) - f(t_i) - c, 0\} + \varepsilon \\
&\leq \alpha \sum_{i=0}^{n-1} \max \{f(t_{i+1}) - f(t_i) - c_1, 0\} \\
&\quad + (1 - \alpha) \sum_{i=0}^{n-1} \max \{f(t_{i+1}) - f(t_i) - c_2, 0\} + \varepsilon \\
&\leq \alpha UTV^{c_1}(f, [a; b]) + (1 - \alpha) UTV^{c_2}(f, [a; b]) + \varepsilon.
\end{aligned}$$

Since  $\varepsilon$  may be arbitrary small, we obtain the convexity assertion. From convexity and monotonicity we obtain the continuity assertion.

The same properties of  $DTV$  and  $TV$  follow immediately from (5.1) and (5.2).

The fact that for  $c \geq \|f\|_{osc}$ ,  $TV^c(f, [a; b]) = 0$  follows easily from equality

$$\max \{|f(t_{i+1}) - f(t_i)| - c, 0\} = 0$$

satisfied for any such  $c$  and  $t_i, t_{i+1} \in [a; b]$ .  $\square$

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